

On construction of boundary preserving numerical schemes

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Abstract

Our aim in this note is to extend the semi discrete technique by combine it with the split step method. We apply our new method to the Ait-Sahalia model and propose an explicit and positivity preserving numerical scheme.

Keywords Explicit numerical scheme, Ait-Sahalia model, boundary preserving.

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1 Introduction

In this paper we describe a technique to construct numerical schemes by combining the split step method (see for example [17]) and the semi discrete method (see [3], [4], [5]). Using the semi discrete method we have constructed explicit and positivity numerical schemes for various stochastic differential equations arising in finance (see [6], [7], [8], [9], [10]).

Using the proposed technique (split-step and semi-discrete) we are able to handle more situations in which we want to construct explicit and boundary preserving numerical schemes. Our starting point was the paper [19] (see also [18]) in which the authors proposed an implicit numerical scheme to approximate the Ait-Sahalia model (see [1]), which is the following,

$$x_t = x_0 + \int_0^t \left(\frac{a_1}{x_s} - a_2 + a_3 x_s - a_4 x_s^r \right) ds + \int_0^t \sigma x_s^\rho dw_s$$

with $x_0 \in \mathbb{R}_+$. We assume that all the coefficients are nonnegative and that $r + 1 > 2\rho$.

Using this new method we describe and analyze a new explicit and positivity preserving numerical scheme for the Ait-Sahalia model which arise in finance. As far as we know this is the first explicit scheme for this model, however this does not mean that from the computational point of view is cheaper than the implicit ones ([19], [18]). We have to make extended numerical experiments in order to compare them.

2 The drift splitting

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a complete probability space with a filtration and let a Wiener process $(W_t)_{t \geq 0}$ defined on this space. Consider the following stochastic differential equation,

$$x_t = x_0 + \int_0^t a(s, x_s) ds + \int_0^t b(s, x_s) dW_s, \quad (1)$$

where $a, b : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable functions and x_0 such that is \mathcal{F}_0 -measurable and square integrable.

Let $0 = t_0 < t_1 < \dots < t_n = T$ and set $\Delta = \frac{T}{n}$. In general, one can split the sde above in m equations. For example, if $a(t, x) = a_1(t, x) + \dots + a_m(t, x)$ then we can have the following splitting

$$\begin{aligned} y_m(0) &= x_0 \\ y_1(t) &= y_m(t_n) + \int_{t_n}^t a_1(s, y_1(s)) ds, \quad t \in (t_n, t_{n+1}] \\ &\vdots \\ y_{m-1}(t) &= y_{m-2}(t_{n+1}) + \int_{t_n}^t a_{m-1}(s, y_{m-1}(s)) ds, \quad t \in (t_n, t_{n+1}] \\ y_m(t) &= y_{m-1}(t_{n+1}) + \int_{t_n}^t a_m(s, y_m(s)) ds + \int_{t_n}^t b(s, y_m(s)) dw_s, \quad t \in (t_n, t_{n+1}] \end{aligned}$$

and approximate each of the above equations by a semi discrete scheme (or another converging scheme).

Then, we can write, for $t \in (t_n, t_{n+1}]$ and $y_m(0) = x_0$

$$\begin{aligned} y_m(t) = y_m(t_n) &+ \int_{t_n}^{t_{n+1}} (a_1(s, y_1(s)) + \dots + a_{m-1}(s, y_{m-1}(s))) ds \\ &+ \int_{t_n}^t a_m(s, y_m(s)) ds + \int_{t_n}^t b(s, y_m(s)) dw_s \end{aligned}$$

or

$$\begin{aligned} y_m(t) = x_0 &+ \int_0^{t_{n+1}} (a_1(s, y_1(s)) + \dots + a_{m-1}(s, y_{m-1}(s))) ds \\ &+ \int_0^t a_m(s, y_m(s)) ds + \int_0^t b(s, y_m(s)) dw_s \end{aligned}$$

We shall denote by $\hat{t} = t_n$ when $t \in [t_n, t_{n+1}]$ and $\tilde{t} = t_{n+1}$ when $t \in [t_n, t_{n+1}]$.

3 Approximating with the semi discrete method

Suppose that there are functions $f_1(t, x, y, z), \dots, f_m(t, x, y, z), f_{m+1}(t, x, y)$ such that $f_i(t, x, x, x) = a_i(t, x)$ for $i = 1, \dots, m$ and $f_{m+1}(t, x, x) = b(t, x)$.

Our numerical scheme depends on the choices of f_i and therefore we should impose conditions on them. For fixed $a(t, x), b(t, x)$ one can choose different f_i in such a way that the corresponding numerical schemes does not converge.

Denoting again our approximation by $y_m(t)$ we write, for $t \in (t_n, t_{n+1}]$

$$\begin{aligned} y_m(0) &= x_0, \\ y_1(t) &= y_m(t_n) + \int_{t_n}^t f_1(s, y_m(t_n), y_1(t), y_1(s)) ds, \\ y_2(t) &= y_1(t_{n+1}) + \int_{t_n}^t f_2(s, y_m(t_n), y_2(t), y_2(s)) ds, \\ &\vdots \\ y_{m-1}(t) &= y_{m-2}(t_{n+1}) + \int_{t_n}^t f_{m-1}(s, y_m(t_n), y_{m-1}(t), y_{m-1}(s)) ds, \\ y_m(t) &= y_{m-1}(t_{n+1}) + \int_{t_n}^t f_m(s, y_m(t_n), y_m(s)) ds + \int_{t_n}^t b(s, y_m(t_n), y_m(s)) dw_s, \end{aligned}$$

It is obvious that we should choose f_i in such a way that all the above equations has at least one strong solution. Then, we have constructed an approximation scheme which is $y_m(t)$ and under

suitable conditions we will show that this converges strongly to the unique strong solution of problem (1). If some of the above equations admits more than one solution then we have constructed at least two approximation schemes and we choose the numerical scheme that has the desired properties, positivity preserving for example.

In a more compact form we can write, for $t \in (t_n, t_{n+1}]$,

$$\begin{aligned} y_m(t) = y_m(t_n) &+ \int_{t_n}^{t_{n+1}} \left(f_1(s, y_1(s), y_1(t), y_m(t_n)) + \dots + f_{m-1}(s, y_{m-1}(s), y_{m-1}(t), y_m(t_n)) \right) ds \\ &+ \int_{t_n}^t f_m(s, y_m(s), y_m(t_n)) ds + \int_{t_n}^t f_{m+1}(s, y_m(s), y_m(t_n)) dw_s, \quad t \in (t_n, t_{n+1}], \quad (2) \end{aligned}$$

with $y_m(0) = x_0$, and also

$$\begin{aligned} y_m(t) = x_0 &+ \int_0^{t_{n+1}} \left(f_1(s, y_1(s), y_1(t), y_m(\hat{s})) + \dots + f_{m-1}(s, y_{m-1}(s), y_{m-1}(t), y_m(\hat{s})) \right) ds \\ &+ \int_0^t f_m(s, y_m(s), y_m(\hat{s})) ds + \int_0^t f_{m+1}(s, y_m(s), y_m(\hat{s})) dw_s, \quad t \in (t_n, t_{n+1}] \quad (3) \end{aligned}$$

Furthermore we have, for $i = 1, \dots, m-1$ and for $t \in (t_n, t_{n+1}]$

$$\begin{aligned} y_i(t) = y_i(t_n) &+ \int_{t_n}^{t_{n+1}} \left(f_1(s, y_1(s), y_1(t), y_m(\hat{s})) + \dots + f_{i-1}(s, y_{i-1}(s), y_{i-1}(t), y_m(\hat{s})) \right) ds \\ &+ \int_{t_{n-1}}^{t_n} \left(f_{i+1}(s, y_{i+1}(s), y_{i+1}(t), y_m(\hat{s})) + \dots + f_m(s, y_m(s), y_m(t), y_m(\hat{s})) \right) ds \\ &+ \int_{t_n}^t f_i(s, y_i(s), y_i(t), y_m(\hat{s})) ds + \int_{t_{n-1}}^{t_n} f_{m+1}(s, y_m(s), y_m(\hat{s})) dw_s, \quad (4) \end{aligned}$$

$$\begin{aligned} y_m(t) = y_i(t_{n+1}) &+ \int_{t_n}^{t_{n+1}} f_i(s, y_i(s), y_i(t), y_m(\hat{s})) + \dots + f_{m-1}(s, y_{m-1}(s), y_{m-1}(t), y_m(\hat{s})) ds \\ &+ \int_{t_n}^t f_m(s, y_m(s), y_m(t), y_m(\hat{s})) ds + \int_{t_n}^t f_{m+1}(s, y_m(s), y_m(\hat{s})) dw_s \quad (5) \end{aligned}$$

Assumption A Suppose that problem (1) has a unique strong solution and that the following moment bounds holds, for every $p > 0$,

$$\mathbb{E}(|y_1(t)|^p + |y_2(t)|^p + \dots + y_m(t) + |x(t)|^p) < \infty$$

Assumption B Suppose that the functions f_i for $i = 1, \dots, m$ satisfy the following locally Lipschitz condition,

$$\begin{aligned} |f_i(t, x, x, x) - f_i(t, x_1, x_2, x_3)| &\leq C_R(|x - x_1| + |x - x_2| + |x - x_3|), \quad i = 2, \dots, m \\ |f_{m+1}(t, x, x) - f_{m+1}(t, x_1, x_2)| &\leq C_R(|x - x_1| + |x - x_2| + |x - x_1|^a) \end{aligned}$$

for $|x_1|, |x_2|, |x_3|, |x| \leq R$, and for some $a \in [\frac{1}{2}, 1)$.

Theorem 1 Under Assumptions A and B we have that

$$\mathbb{E}|y_m(t) - x(t)|^2 \leq C_R \Delta + \frac{2^{p+1} \delta A}{p} + \frac{(p-2)2A}{p \delta^{\frac{2}{p-2}} R^p} + e_{q-1} + \frac{C_R \Delta}{q e_q}$$

where $e_q = e^{-q(q+1)/2}$ for every $q \in \mathbb{N}$. Therefore, for every $\varepsilon > 0$, we can fix first big enough q , then small enough δ and big enough R and finally for small enough Δ we obtain that $\mathbb{E}|y_m(t) - x(t)|^2 \leq \varepsilon$.

Proof. Set $\rho_R = \inf\{t \in [0, T] : |x(t)| \geq R\}$, $\tau_R^i = \inf\{t \in [0, T] : |y_i(t)| \geq R\}$ for $i = 1, \dots, m$. Let $\theta_R = \min\{\tau_R^i, \rho_R\}$.

We can prove that $\mathbb{P}(\tau_r^i \leq T \text{ or } \rho_R \leq T) \leq \frac{2A}{R^p}$. Using Young inequality we obtain, for any $\delta > 0$,

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} |y_m(t) - x(t)|^2 \right) \leq \mathbb{E} \left(\sup_{0 \leq t \leq T} |y_m(t \wedge \theta_R) - x(t \wedge \theta_R)|^2 \right) + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)2A}{p\delta^{\frac{2}{p-2}} R^p}.$$

The difference $x(t) - y_m(t)$ is as follows,

$$\begin{aligned} x(t) - y_m(t) = & \int_0^t \sum_{i=1}^m (f_i(s, x(s), x(s), x(s)) - f_i(s, y_i(s), y_i(t), y_m(\hat{s}))) ds \\ & + \int_0^t (f_{m+1}(s, x(s), x(s)) - f_{m+1}(s, y_m(s), y_m(\hat{s}))) dw_s \\ & + \int_t^{t_{n+1}} \sum_{i=1}^{m-1} (f_i(s, x(s), x(s), x(s)) - f_i(s, y_i(s), y_i(t), y_m(\hat{s}))) ds \end{aligned}$$

We shall estimate the term $|x(t \wedge \theta_R) - y_m(t \wedge \theta_R)|^2$ as follows, using Assumption B

$$\begin{aligned} & \mathbb{E}|x(t \wedge \theta_R) - y_m(t \wedge \theta_R)|^2 \\ \leq & C_R \Delta^2 + C_R \int_0^{t \wedge \theta_R} \sum_{i=1}^m \mathbb{E}(|x(s) - y_m(s)|^2 + |y_m(s) - y_i(\tilde{s})|^2 \\ & + |y_m(s) - y_m(\hat{s})|^2 + |y_i(\tilde{s}) - y_i(s)|^2 + |x(s) - y_m(s)|^{2a}) \\ \leq & C_R \sqrt{\Delta} + C_R \int_0^{t \wedge \theta_R} (\mathbb{E}|x(s) - y_m(s)|^2 + \mathbb{E}|x(s) - y_m(s)|) ds \end{aligned} \quad (6)$$

The term $C_R \Delta^2$ comes from the estimation of

$$\int_t^{t_{n+1}} \sum_{i=1}^{m-1} (f_i(s, x(s), x(s), x(s)) - f_i(s, y_i(s), y_i(t), y_m(\hat{s}))) ds$$

To get (6) above we have used the following, for $i = 1, \dots, m+1$

$$\begin{aligned} |x(s) - y_i(s)| & \leq |x(s) - y_m(s)| + |y_m(s) - y_i(s)| \\ & \leq |x(s) - y_m(s)| + |y_m(s) - y_i(\tilde{s})| + |y_i(\tilde{s}) - y_i(s)| \end{aligned}$$

combined with (4) and (5). Furthermore we have used that for $2a \geq 1$ it holds that

$$\mathbb{E}|x(s) - y_i(s)|^{2a} = \mathbb{E}|x(s) - y_i(s)||x(s) - y_i(s)|^{2a-1} \leq C_R \mathbb{E}|x(s) - y_i(s)|$$

From (4) and (5) we have, for $s \in (t_n, t_{n+1}]$ and $i = 1, \dots, m$,

$$\begin{aligned} \mathbb{E}|y_i(s \wedge \theta_R) - y_i(t_n \wedge \theta_R)|^2 & \leq C_R \Delta, \\ \mathbb{E}|y_m(s \wedge \theta_R) - y_i(t_{n+1} \wedge \theta_R)|^2 & \leq C_R \Delta, \\ \mathbb{E}|y_i(t_{n+1} \wedge \theta_R) - y_i(s \wedge \theta_R)|^2 & \leq C_R \Delta \end{aligned}$$

and all these estimates are useful to get (6).

We should estimate the following quantity (and substitute this estimation to (6)),

$$\mathbb{E}|x(s) - y_m(s)|$$

Let the non increasing sequence $\{e_q\}_{q \in \mathbb{N}}$ with $e_q = e^{-q(q+1)/2}$ and $e_0 = 1$. We introduce the following sequence of smooth approximations of $|x|$,

$$\phi_q(x) = \int_0^{|x|} dy \int_0^y \psi_q(u) du,$$

where the existence of the continuous function $\psi_q(u)$ with $0 \leq \psi_q(u) \leq 2/(qu)$ and support in (e_q, e_{q-1}) is justified by $\int_{e_q}^{e_{q-1}} (du/u) = q$. The following relations hold for $\phi_q \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ with $\phi_q(0) = 0$,

$$|x| - e_{q-1} \leq \phi_q(x) \leq |x|, \quad |\phi_q'(x)| \leq 1, \quad x \in \mathbb{R},$$

$$|\phi_q''(x)| \leq \frac{2}{q|x|}, \quad \text{when } e_q < |x| < e_{q-1} \quad \text{and} \quad |\phi_q''(x)| = 0 \quad \text{otherwise.}$$

Applying Ito's formula on $\phi_q(x(t) - y_m(t))$ for $t \in [0, t \wedge \theta_R]$ we get

$$\begin{aligned} \mathbb{E}\phi_q(x(t) - y_m(t)) &\leq C_R\Delta + \frac{C_R\Delta}{qe_q} + C_R \int_0^t \mathbb{E}\phi_q'(x(t) - y_m(t))|x(s) - y_m(s)|ds \\ &\quad + C_R \int_0^t \mathbb{E}|x(s) - y_m(s)|ds \end{aligned}$$

Therefore

$$\mathbb{E}|x(t) - y_m(t)| \leq e_{q-1} + C_R\Delta + \frac{C_R\Delta}{qe_q} + C_R \int_0^t \mathbb{E}|x(s) - y_m(s)|ds$$

Applying Gronwall's inequality and substituting in (6) and then again Gronwall inequality we get the desired result. \square

An example, is the following stochastic differential equation,

$$x_t = x_0 + \int_0^t k(l - x_s) - dx_s^2 ds + \sigma \int_0^t \sqrt{x_s} dw_s$$

For this sde, we propose the following splitting, for $t \in (t_n, t_{n+1}]$

$$\begin{aligned} y_2(0) &= x_0, \\ y_1(t) &= y_2(t_n) + \int_{t_n}^t -ky_1(s) - dy_1^2(s)ds, \\ y_2(t) &= y_1(t_{n+1}) + \int_{t_n}^t klds + \sigma \int_{t_n}^t \sqrt{y_2(s)}dw_s \end{aligned}$$

The first equation can be approximated as follows, denoting again by $y_1(t)$ the approximation

$$y_1(t) = y_2(t_n) + \int_{t_n}^t -ky_1(s) - dy_1(s)y_2(\hat{s})ds$$

which produces a positive solution whenever $y_2(t_n) > 0$. The second equation can be approximated in the spirit of [4].

4 Application to the Ait-Sahalia model

In the previous section we have described a new technique to construct numerical schemes by combining the splitting technique and the semi discrete method. We have proved a convergence result when the numerical scheme satisfy some classic hypotheses. Below we shall use this technique to construct an explicit and positivity preserving numerical scheme for the Ait-Sahalia model which is the following,

$$x_t = x_0 + \int_0^t \frac{1}{2} \left(\frac{a_1}{x_s} - a_2 + a_3x_s - a_4x_s^r \right) ds + \int_0^t \sigma x_s^\rho dw_s$$

with $x_0 \in \mathbb{R}_+$. We first use the transformation

$$y_t = x_t^2$$

By using Ito's formula we obtain

$$y(t) = y(0) + \int_0^t \left(a_1 - a_2 \sqrt{y(s)} + a_3 y(s) - a_4 y^{\frac{r+1}{2}}(s) + \sigma^2 y^\rho(s) \right) ds + \int_0^t 2\sigma y^{\frac{\rho+1}{2}}(s) dw_s$$

We assume that that $r+1 > 2\rho$. Then, we split as follows, introducing a free parameter $a > 0$,

$$\begin{aligned} y_2(0) &= x_0^2 \\ y_1(t) &= y_2(t_n) + \int_{t_n}^t (a y_1(s) - a_2 \sqrt{y_1(s)}) ds, \\ y_2(t) &= y_1(t_{n+1}) + \int_{t_n}^t (a_1 + (a_3 - a) y_2(s) - a_4 y_2^{\frac{r+1}{2}}(s) + \sigma^2 y_2^\rho(s)) ds + \int_{t_n}^t 2\sigma y_2^{\frac{\rho+1}{2}}(s) dw_s \end{aligned} \quad (7)$$

It is easy to verify that equation (7) has at least one solution in each interval and one of them is the following

$$y_1(t) = \left(\frac{a_2}{a} + (\sqrt{y_2(t_n)} - \frac{a_2}{a}) e^{a \frac{(t-t_n)}{2}} \right)^2$$

which is always positive and well posed whenever $y_2(t_n) \geq 0$.

We can approximate equation (8) by using a semi discrete approach, namely

$$y_2(t) = y_1(t_{n+1}) + \int_{t_n}^t a_1 + y_2(s) (a_3 - a + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s})) ds + \int_{t_n}^t 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s$$

which have a positive and known strong solution whenever $y_1(t_{n+1}) \geq 0$. We denote again by $y_2(t)$ the approximation of (8).

We will use later on the following forms of y_1, y_2 , for $t \in (t_n, t_{n+1}]$,

$$\begin{aligned} y_1(t) &= y_1(t_n) + \int_{t_{n-1}}^{t_n} a_1 + y_2(s) (a_3 - a + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s})) ds \\ &\quad + \int_{t_{n-1}}^{t_n} 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s + \int_{t_n}^t (a y_1(s) - a_2 \sqrt{y_1(s)}) ds, \\ y_2(t) &= y_2(t_n) + \int_{t_n}^t (a y_1(s) - a_2 \sqrt{y_1(s)} + a_1 + y_2(s) (a_3 - a + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))) ds \\ &\quad + \int_{t_n}^t 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s + \int_t^{t_{n+1}} a y_1(s) - a_2 \sqrt{y_1(s)} ds \end{aligned}$$

Proposition 1 *If $r+1 > 2\rho$ then we have the following moment bounds, for $\Delta < 1$ if $a = \ln \frac{4}{3}$,*

$$\mathbb{E}(|y_1(t)|^p + |y_2(t)|^p) < \infty$$

Proof. Let $\tau_R = \inf\{t \in [0, T] : |y_2(t)| > R\}$. Note that $y_1(t \wedge \tau_R)$ is also uniformly bounded for all $\omega \in \Omega$.

We can write

$$\begin{aligned} y_2(t \wedge \tau_R) &= y_2(0) + \int_0^{t \wedge \tau_R} (a_1 - a_2 \sqrt{y_1(s)} + a y_1(s) + y_2(s) (a_3 - a + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))) ds \\ &\quad + \int_0^{t \wedge \tau_R} 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s + \int_{t \wedge \tau_R}^{t_{n+1} \wedge \tau_R} a y_1(s) - a_2 \sqrt{y_1(s)} ds \end{aligned}$$

But

$$\begin{aligned} y_2(t \wedge \tau_R) \leq y_2(0) &+ \int_0^{t \wedge \tau_R} (a_1 + ay_1(s) + y_2(s)(a_3 + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))) ds \\ &+ \int_0^{t \wedge \tau_R} 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s + \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} ay_1(s) ds \end{aligned}$$

Let us estimate

$$\begin{aligned} \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} ay_1(s) ds &= a \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} \left(\frac{a_2}{a} + (\sqrt{y_2(t_n)} - \frac{a_2}{a}) e^{a \frac{(t-t_n)}{2}} \right)^2 ds \\ &= a \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} \left(\frac{a_2^2}{a^2} + (\sqrt{y_2(t_n)} - \frac{a_2}{a})^2 e^{a(t-t_n)} + 2 \frac{a_2}{a} (\sqrt{y_2(t_n)} - \frac{a_2}{a}) e^{a \frac{(t-t_n)}{2}} \right) ds \\ &\leq C + \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} a (\sqrt{y_2(t_n)} - \frac{a_2}{a})^2 e^{a(t-t_n)} + 2a \frac{a_2^2}{a^2} + \frac{1}{2} a (\sqrt{y_2(t_n)} - \frac{a_2}{a})^2 e^{a(t-t_n)} ds \\ &= C + \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} \frac{3}{2} a (\sqrt{y_2(t_n)} - \frac{a_2}{a})^2 e^{a(t-t_n)} ds \\ &\leq C + \frac{3a_2^2 \Delta}{2a} e^{a\Delta} + \int_{t_n \wedge \tau_R}^{t_{n+1} \wedge \tau_R} \frac{3}{2} ay_2(t_n) e^{a(t-t_n)} ds \\ &\leq C + \frac{3}{2} y_2(t_n \wedge \tau_R) (e^{a\Delta} - 1) \end{aligned}$$

Therefore

$$\begin{aligned} y_2(t \wedge \tau_R) \leq C &+ \frac{3}{2} y_2(t_n \wedge \tau_R) (e^{a\Delta} - 1) + \int_0^{t \wedge \tau_R} (a_1 + ay_1(s) + y_2(s)(a_3 + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))) ds \\ &+ \int_0^{t \wedge \tau_R} 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s \end{aligned}$$

Denoting by $v(s \wedge \tau_R)$ the following Ito process,

$$\begin{aligned} v(s \wedge \tau_R) = C &+ \frac{3}{2} y_2(t_n \wedge \tau_R) (e^{a\Delta} - 1) + \int_0^{t \wedge \tau_R} (a_1 + ay_1(s) + y_2(s)(a_3 + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))) ds \\ &+ \int_0^{t \wedge \tau_R} 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s \end{aligned}$$

we see that $y_2(s \wedge \tau_R) \leq v(s \wedge \tau_R)$. We shall prove that $v(s)$ has bounded p th moments and therefore $y_2(t)$ also.

Applying Ito's formula on $v^p(t \wedge \tau_R)$ we get that

$$\begin{aligned} v^p(t \wedge \tau_R) = & (C + \frac{3}{2} y_2(t_n \wedge \tau_R) (e^{a\Delta} - 1))^p \\ &+ \int_0^{t \wedge \tau_R} p v^{p-1} (a_1 + ay_1(s) + y_2(s)(a_3 + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))) ds \\ &+ \int_0^{t \wedge \tau_R} \frac{p(p-1)}{2} 4\sigma^2 y_2^2(s) v^{p-2}(s) y_2^{\rho-1}(\hat{s}) ds \\ &+ \int_0^{t \wedge \tau_R} p 2\sigma y_2(s) v^{p-1}(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s \end{aligned}$$

Taking expectations we arrive at

$$\begin{aligned} \mathbb{E}v^p(t \wedge \tau_R) \leq & \mathbb{E}(C + \frac{3}{2} y_2(t_n \wedge \tau_R) (e^{a\Delta} - 1))^p \\ &+ \int_0^{t \wedge \tau_R} (C + C\mathbb{E}v^p(s) + pa\mathbb{E}y_1(s)v^{p-1}(s)) ds \\ &+ \int_0^{t \wedge \tau_R} p\mathbb{E}v^p(s)(a_3 + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s})) + \frac{p(p-1)}{2} 4\sigma^2 y_2^{\rho-1}(\hat{s}) ds \end{aligned}$$

We have assumed that $r + 1 > 2\rho$ so there exists some constant C independent of ω, Δ such that

$$a_3 + \sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}) + \frac{p(p-1)}{2} 4\sigma^2 y_2^{\rho-1}(\hat{s}) \leq C$$

We will estimate now the term, using the inequality $(a+b)^p \leq 2^{p-1}a^p + 2^{p-1}b^p$,

$$\mathbb{E}(C + \frac{3}{2}y_2(t_n \wedge \tau_R)(e^{a\Delta} - 1))^p \leq C + \frac{3^p}{2}(e^{a\Delta} - 1)^p \mathbb{E}y_2^p(t_n \wedge \tau_R)$$

Furthermore the following term

$$\mathbb{E}y_1(s)v^{p-1}(s) = \mathbb{E}\left(\frac{a_2}{a} + (\sqrt{y_2(\hat{s})} - \frac{a_2}{a})e^{a\frac{(t-t_n)}{2}}\right)^2 v^{p-1}(s) \leq C + C \sup_{0 \leq l \leq s} \mathbb{E}v^p(l)$$

Collecting all the above results, we obtain

$$\mathbb{E}v^p(t \wedge \tau_R) \leq C + \frac{3^p}{2}(e^{a\Delta} - 1)^p \mathbb{E}v^p(t_n \wedge \tau_R) + C \int_0^{t \wedge \tau_R} \mathbb{E}v^p(s) + \sup_{0 \leq l \leq s} \mathbb{E}v^p(l) ds$$

Next setting

$$u(s) = \sup_{0 \leq l \leq s} \mathbb{E}v^p(l)$$

we can write

$$(1 - \frac{3^p}{2}(e^{a\Delta} - 1)^p)u(t) \leq C + \int_0^{t \wedge \tau_R} Cu(s) ds$$

Now it is the time to choose accordingly the free parameter a so as

$$\frac{3^p}{2}(e^{a\Delta} - 1)^p < 1$$

Choosing $a = \ln \frac{4}{3}$ we easily see that the above inequality holds for every $\Delta < 1$. It is clear that if we use smaller Δ then we can choose bigger a so that the corresponding constants will be smaller.

Using Gronwall's inequality and then Fatou's lemma we get the result. \square

Unfortunately, we can not use our Theorem 1 directly to get the desired result, therefore we will argue differently.

Theorem 2 *If $r + 1 > 2\rho$ then*

$$\mathbb{E}|y(t) - y_2(t)|^2 \leq C_R \Delta + \frac{C}{R}$$

for any $R > 0$. Therefore, for every $\varepsilon > 0$ we fix R such that $\frac{C}{R} < \varepsilon$ and then for small enough Δ we have that

$$\mathbb{E}|y(t) - y_2(t)|^2 \leq \varepsilon$$

Proof. The approximate solution $y_2(t)$ is as follows,

$$\begin{aligned} y_2(t) = x_0^2 &+ \int_0^t (a_1 - a_2 \sqrt{y_1(s)} + (a_3 - a)y_2(s) + ay_1(s) - a_4 y_2(s)y_2^{\frac{r-1}{2}}(\hat{s}) + \sigma^2 y_2(s)y_2^{\rho-1}(\hat{s})) ds \\ &+ \int_0^t 2\sigma y_2(s)y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s + \int_t^{t_{n+1}} ay_1(s) - a_2 \sqrt{y_1(s)} ds \end{aligned}$$

Set

$$\begin{aligned} v(t) = x_0^2 &+ \int_0^t (a_1 - a_2 \sqrt{y_1(s)} + (a_3 - a)y_2(s) + ay_1(s) - a_4 y_2(s) y_2^{\frac{r-1}{2}}(\hat{s}) + \sigma^2 y_2(s) y_2^{\rho-1}(\hat{s})) ds \\ &+ \int_0^t 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}) dw_s \end{aligned}$$

Then it is clear that $\mathbb{E}|y_2(t) - v(t)|^p \leq C\Delta^p$ and therefore $v(t)$ has bounded moments as well.

The difference $y(t) - v(t)$ is as follows

$$\begin{aligned} y(t) - v(t) &= \int_0^t \left(a_2(\sqrt{y_1(s)} - \sqrt{y(s)}) + (a_3 - a)(y(s) - y_2(s)) + a(y(s) - y_1(s)) \right. \\ &\quad \left. + a_4(y_2(s) y_2^{\frac{r-1}{2}}(\hat{s}) - y^{\frac{r+1}{2}}(s) + \sigma^2(y^\rho(s) - y_2(s) y_2^{\rho-1}(\hat{s}))) \right) ds \\ &\quad + \int_0^t 2\sigma(y^{\frac{\rho+1}{2}}(s) - y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s})) dw_s \end{aligned}$$

Applying Ito's formula on $(y(t) - v(t))^2$ and setting

$$\begin{aligned} g(t) &= a_2(\sqrt{y_1(s)} - \sqrt{y(s)}) + (a_3 - a)(y(s) - y_2(s)) + a(y(s) - y_1(s)) \\ &\quad + a_4(y_2(s) y_2^{\frac{r-1}{2}}(\hat{s}) - y^{\frac{r+1}{2}}(s) + \sigma^2(y^\rho(s) - y_2(s) y_2^{\rho-1}(\hat{s}))) \end{aligned}$$

we obtain

$$\mathbb{E}(y(t) - v(t))^2 = \mathbb{E} \int_0^t 2(y(t) - y_2(t))g(s) + 2(y_2(t) - v(t))g(s) + 4\sigma^2(y^{\frac{\rho+1}{2}}(s) - y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s}))^2 ds$$

Then, it is clear that

$$2\mathbb{E}(y_2(t) - v(t))g(s) \leq C\Delta$$

using the moment bounds of y, y_2 .

Next we will estimate the term

$$\begin{aligned} 2a_2\mathbb{E}(y(t) - y_2(t))(\sqrt{y_1(s)} - \sqrt{y(s)}) &= 2a_2\mathbb{E}(y(t) - y_2(t))(\sqrt{y_2(s)} - \sqrt{y(s)}) \\ &\quad + 2a_2\mathbb{E}(y(t) - y_2(t))(\sqrt{y_1(s)} - \sqrt{y_2(s)}) \\ &\leq 2a_2\mathbb{E}(y(t) - y_2(t))(\sqrt{y_1(s)} - \sqrt{y_2(s)}) \end{aligned}$$

where we have used the fact that $h(x) = -\sqrt{x}$ is a decreasing function. Using Holder's inequality and the fact that y, y_1, y_2 have bounded moments, we obtain

$$\mathbb{E}(y(t) - y_2(t))(\sqrt{y_1(s)} - \sqrt{y_2(s)}) \leq \sqrt{\mathbb{E}|y(t) - y_2(t)|^2} \sqrt{\mathbb{E}|y_1(s) - y_2(s)|} \leq C \sqrt{\mathbb{E}|y_1(s) - y_2(s)|}$$

Therefore it remains to estimate the term

$$\begin{aligned} \mathbb{E}|y_1(s) - y_2(s)| &\leq \mathbb{E}|y_1(\tilde{s}) - y_2(s)| + \mathbb{E}|y_1(s) - y_1(\tilde{s})| \\ &\leq \mathbb{E}|y_1(\tilde{s}) - y_2(s)| + \mathbb{E}|y_1(\tilde{s}) - y_1(\hat{s})| + \mathbb{E}|y_1(\hat{s}) - y_1(s)| \\ &\leq C\sqrt{\Delta} \end{aligned}$$

Now we will estimate the term

$$\begin{aligned} &2\mathbb{E}(y(t) - y_2(t)) \left((a_3 - a)(y(s) - y_2(s)) + a(y(s) - y_1(s)) \right) \\ &= 2a_3\mathbb{E}(y(t) - y_2(t))^2 + 2a\mathbb{E}(y(t) - y_2(t))(y_1(t) - y_2(t)) \\ &\leq C\mathbb{E}(y(t) - y_2(t))^2 + C\sqrt{\Delta} \end{aligned}$$

The term

$$\begin{aligned}
& 2\mathbb{E}(y(t) - y_2(t))(a_4(y_2(s)y_2^{\frac{r-1}{2}}(\hat{s}) - y_2^{\frac{r+1}{2}}(s))) \\
&= 2a_4\mathbb{E}(y(t) - y_2(t))(y_2^{\frac{r+1}{2}}(s) - y_2^{\frac{r+1}{2}}(\hat{s})) + 2a_4\mathbb{E}(y(t) - y_2(t))y_2(s)(y_2^{\frac{r-1}{2}}(\hat{s}) - y_2^{\frac{r-1}{2}}(s)) \\
&\leq C\Delta + 2a_4\mathbb{E}(y(t) - y_2(t))y_2(s)(y_2^{\frac{r-1}{2}}(\hat{s}) - y_2^{\frac{r-1}{2}}(s))
\end{aligned}$$

using the fact that $h(x) = -x^{\frac{r+1}{2}}$ is a decreasing function. But

$$\begin{aligned}
& 2a_4\mathbb{E}(y(t) - y_2(t))y_2(s)(y_2^{\frac{r-1}{2}}(\hat{s}) - y_2^{\frac{r-1}{2}}(s)) \\
&= 2a_4\mathbb{E}(y(t) - y_2(t))(y_2^{\frac{r+1}{2}}(\hat{s}) - y_2^{\frac{r+1}{2}}(s)) + 2a_4\mathbb{E}(y(t) - y_2(t))(y_2(s) - y_2(\hat{s}))y_2^{\frac{r-1}{2}}(\hat{s})
\end{aligned}$$

Since $\mathbb{E}y_2^{\frac{r-1}{2}}(t) < \infty$ we can use the mean value theorem arriving to the following estimate,

$$\mathbb{E}(y(t) - y_2(t))(a_4(y_2(s)y_2^{\frac{r-1}{2}}(\hat{s}) - y_2^{\frac{r+1}{2}}(s))) \leq C\sqrt{\Delta}$$

The term

$$2\mathbb{E}(y(t) - y_2(t))(y^\rho(t) - y_2^\rho(t)) = 2(\rho - 1)\mathbb{E}(y(t) - y_2(t))^2 h^{\rho-1}(t)$$

using the mean value theorem, for some $h(t)$ located between $y(t), y_2(t)$. Using the moment bounds of y, y_2 we arrive at, for any $R > 0$,

$$\begin{aligned}
2(\rho - 1)\mathbb{E}(y(t) - y_2(t))^2 h^{\rho-1}(t) &= 2(\rho - 1)\mathbb{E}(y(t) - y_2(t))^2 h^{\rho-1}(t) \mathbb{I}_{\{h^{\rho-1}(t) > R\}} \\
&\quad + 2(\rho - 1)\mathbb{E}(y(t) - y_2(t))^2 h^{\rho-1}(t) \mathbb{I}_{\{h^{\rho-1}(t) \leq R\}} \\
&\leq C_R \mathbb{E}|y(t) - y_2(t)|^2 + 2(\rho - 1)\mathbb{E}(y(t) - y_2(t))^2 h^{\rho-1}(t) \mathbb{I}_{\{h^{\rho-1}(t) > R\}}
\end{aligned}$$

Using Holder's inequality we deduce that

$$\mathbb{E}(y(t) - y_2(t))^2 h^{\rho-1}(t) \mathbb{I}_{\{h^{\rho-1}(t) > R\}} \leq \sqrt{\mathbb{E}(y(t) - y_2(t))^4 h^{2\rho-2}(t)} \sqrt{\mathbb{E} \mathbb{I}_{\{h^{\rho-1}(t) > R\}}}$$

Using the moment bounds of y, y_2, h and Markov's inequality we arrive at the following estimate,

$$2\mathbb{E}(y(t) - y_2(t))(y^\rho(t) - y_2^\rho(t)) \leq C_R \mathbb{E}|y(t) - y_2(t)|^2 + \frac{C}{R}$$

The same holds for the term $4\sigma^2\mathbb{E}(y^{\frac{\rho+1}{2}}(s) - y_2(s)y_2^{\frac{\rho-1}{2}}(\hat{s}))^2$ therefore we conclude that

$$\mathbb{E}(y(t) - v(t))^2 \leq \frac{C}{R} + C\sqrt{\Delta} + C_R \int_0^t \mathbb{E}(y(s) - v(s))^2 ds$$

and using Gronwall's inequality we conclude the desired result.

Now it is easy to see that

$$\mathbb{E}|y(t) - y_2(t)|^2 \leq C\mathbb{E}|y(t) - v(t)|^2 + C\mathbb{E}|y_2(t) - v(t)|^2 \rightarrow 0 \text{ as } \Delta \rightarrow 0$$

□

Theorem 3 *If $r + 1 > 2\rho$ we have that*

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|\sqrt{y_2(t)} - \sqrt{y(t)}|^2 = 0$$

Proof. Using Holder's inequality and the nonnegativity of the approximation we have

$$\begin{aligned}\mathbb{E}|\sqrt{y_2(t)} - \sqrt{y(t)}|^2 &= \mathbb{E}\frac{|y_2(t) - y(t)|^2}{(\sqrt{y(t)} + \sqrt{y_2(t)})^2} \\ &\leq \sqrt{\mathbb{E}|y_2(t) - y(t)|^2}^4 \sqrt{\mathbb{E}|y(t) + y_2(t)|^4 \mathbb{E}\frac{1}{y^2(t)}}\end{aligned}$$

Recalling from [19] that the inverse moments of the true solution are bounded we conclude our proof. \square

In practice, we will approximate equation (8) with a slightly different approximation which is the following

$$y_2(t) = y_1(t_{n+1}) + a_1\Delta + \int_{t_n}^t y_2(s)(\sigma^2 y_2^{\rho-1}(\hat{s}) - a_4 y_2^{\frac{r-1}{2}}(\hat{s}))ds + \int_{t_n}^t 2\sigma y_2(s) y_2^{\frac{\rho-1}{2}}(\hat{s})dw_s$$

Our numerical scheme for the transformed Ait-Sahalia model is as follows

$$\begin{aligned}y_0 &= x_0^2, \\ y_{n+1} &= \left(a_1\Delta + \left(\frac{a_2}{a} + (\sqrt{y_n} - \frac{a_2}{a})e^{\frac{a\Delta}{2}}\right)^2\right)e^{-\Delta(\sigma^2 y_n^{\rho-1} + a_4 y_n^{\frac{r-1}{2}}) + 2\sigma y_n^{\frac{\rho-1}{2}}\Delta W}\end{aligned}$$

The approximation of the Ait-Sahalia model will be $x_n = \sqrt{y_n}$.

5 Summary and comments

In this paper we extend the semi discrete method by combine it with the split step method. We can, in general, split our stochastic differential equation in m equations (by splitting the drift term) and then in each equation we apply any approximation method. The aim of this new technique is that the resulting numerical scheme will be boundary preserving.

Consider for example the following Ait-Sahalia type model,

$$x_t = x_0 + \int_0^t \frac{1}{2} \left(\frac{a_1}{x_s} - a_2 - b_1 \sqrt{x_s} - b_2 x_s^{\frac{3}{2}} - b_3 \ln(1 + x_s^2) + a_3 x_s - a_4 x_s^r \right) ds + \int_0^t \sigma x_s^\rho dw_s.$$

Using the transformation $y_t = x_t^2$ we get the following stochastic differential equation for y_t ,

$$\begin{aligned}y_t = y_0 &+ \int_0^t \left(a_1 - a_2 \sqrt{y_s} - b_1 y_s^{\frac{3}{4}} - b_2 y_s^{\frac{5}{4}} - b_3 \sqrt{y_s} \ln(1 + y_s) + a_3 y_s - a_4 y_s^{\frac{r+1}{2}} + \sigma^2 y_s^\rho \right) ds \\ &+ \int_0^t 2\sigma y_s^{\frac{\rho+1}{2}} dw_s\end{aligned}$$

Then we propose the following slit step combined with the semi discrete technique numerical scheme, for $t \in (t_n, t_{n+1}]$

$$\begin{aligned}y_5(0) &= x_0, \\ y_1(t) &= y_5(t_n) - b_3 \int_{t_n}^t \sqrt{y_1(s)} \ln(1 + y_5(t_n)) ds, \\ y_2(t) &= y_1(t_{n+1}) - b_2 \int_{t_n}^t y_2(s) y_5^{\frac{1}{4}}(t_n) ds, \\ y_3(t) &= y_2(t_{n+1}) - b_1 \int_{t_n}^t y_3^{\frac{3}{4}}(s) ds, \\ y_4(t) &= y_3(t_{n+1}) + \int_{t_n}^t a_3 y_4(s) - a_2 \sqrt{y_4(s)} ds, \\ y_5(t) &= y_4(t_{n+1}) + a_1\Delta + \int_{t_n}^t y_5(s) \left(-a_4 y_5^{\frac{r-1}{2}}(t_n) + \sigma^2 y_5^{\rho-1}(t_n) \right) ds + \int_{t_n}^t 2\sigma y_5^{\frac{\rho-1}{2}}(t_n) y_5(s) dw_s\end{aligned}$$

The solutions are

$$\begin{aligned}
y_1(t) &= \left(\sqrt{y_5(t_n)} - \frac{b_3 \ln(1 + y_5(t_n))}{2} (t - t_n) \right)^2, \\
y_2(t) &= y_1(t_{n+1}) e^{-b_2 y_5(t_n)(t-t_n)}, \\
y_3(t) &= \left(\sqrt[4]{y_2(t_{n+1})} - b_1 \frac{t - t_n}{4} \right)^4, \\
y_4(t) &= \left(\frac{a_2}{a_3} + \left(\sqrt{y_3(t_n)} - \frac{a_2}{a_3} \right) e^{a_3 \frac{(t-t_n)}{2}} \right)^2, \\
y_5(t) &= (y_4(t_{n+1}) + a_1 \Delta) e^{-\Delta(\sigma^2 y_5^{\rho-1}(t_n) + a_4 y_5^{\frac{\rho-1}{2}}(t_n)) + 2\sigma y_5^{\frac{\rho-1}{2}}(t_n) \Delta W}
\end{aligned}$$

This use of the splitting-semi discrete technique produces a positivity preserving and explicit numerical scheme.

Another, obviously generalization, is that we can semi-discretize in the time variable also. For example we can assume the following assumption,

Assumption C Suppose that the functions f_i for $i = 1, \dots, m$ satisfy the following locally Lipschitz condition,

$$\begin{aligned}
|f_i(t, t, x, x, x) - f_i(t, t_1, x_1, x_2, x_3)| &\leq C_R(|t - t_1|^{b_1} + |x - x_1| + |x - x_2| + |x - x_3|), \quad i = 2, \dots, m \\
|f_{m+1}(t, t, x, x) - f_{m+1}(t, t_1, x_1, x_2)| &\leq C_R(|t - t_1|^{b_2} + |x - x_1| + |x - x_2| + |x - x_1|^a)
\end{aligned}$$

for $|x_1|, |x_2|, |x_3|, |x| \leq R$, for $t, t_1 \in [0, T]$, for some $a \in [\frac{1}{2}, 1)$ and for some $b_1, b_2 > 0$. With this kind of generalization we can handle problems which has time depending parameters (see [3]).

There is also the possibility to split the diffusion term. For example, if $a_1 + \dots + a_m = a$ and $b_1 + \dots + b_l = b$ then for $t \in (t_n, t_{n+1}]$,

$$\begin{aligned}
y_m(0) &= x_0, \\
y_1(t) &= y_m(t_n) + \int_{t_n}^t a_1(s, y_1(s)) ds, \\
&\vdots \\
y_i(t) &= y_{i-1}(t_{n+1}) + \int_{t_n}^t a_i(s, y_i(s)) ds + \int_{t_n}^t b_1(s, y_i(s)) dw_s, \\
&\vdots \\
y_j(t) &= y_{j-1}(t_{n+1}) + \int_{t_n}^t a_j(s, y_j(s)) ds, \\
&\vdots \\
y_m(t) &= y_m(t_{n+1}) + \int_{t_n}^t a_m(s, y_2(s)) ds + \int_{t_n}^t b_l(s, y_m(s)) dw_s
\end{aligned}$$

where $i, j \in \{1, \dots, m\}$. The general idea is that if we want to construct a numerical scheme with values in a domain D then we can split accordingly such as $y_1 \in D_1$, whenever the initial value (for y_1) is in D , $y_2 \in D_2$ whenever the initial value is in D_1 and finally $y_m \in D$ whenever the initial value for y_m is in D_{m-1} .

To approximate any of the above equations which do not have a diffusion term we can use any suitable numerical scheme and any semi discrete approximation. The same holds also for the first stochastic differential equation (in our setting is the i -equation) but to approximate any other stochastic differential equation (i.e. with a diffusion term) we should fully discretize the corresponding sde, i.e. we can not use a semi discrete method. If we want to produce a boundary preserving numerical scheme then these sdes (that we should fully discretize) can be approximated by balanced Milstein methods (see [14], [15], [16] and [2]).

An interesting question (that we do not answer in this paper) is the rate of convergence of the explicit numerical schemes that the semi discrete method produces. In [11], [12], [13] the authors study the convergence rates of various numerical schemes and it seems that these techniques will be useful also for the semi discrete schemes.

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